

Fourier transform method for the electrostatic self-energy of a solid sphere with uniform volume charge density

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The problem of a solid sphere with uniform volume charge density is encountered in virtually all undergraduate calculus-based physics textbooks dealing with the topic of electromagnetism. This example illustrates well the use of Gauss's law and from there one can easily derive all the quantities of interest such as electrostatic field, potential, self-energy, and so on. Undergraduate physics majors are also well aware of the theory of Fourier transforms from having taken mathematics courses. Nevertheless, despite its great utility, the Fourier transform method is rarely mentioned as a powerful tool to solve physics problems at this level. To address this shortcoming, in this work we propose a possible scenario which may allow an instructor to introduce this powerful method to a proper undergraduate audience without any major pedagogical drawback. The case study that we choose is that of a solid sphere with uniform volume charge density. Specifically, we show the calculation of its electrostatic self-energy by using Fourier transform techniques. The main idea of this work is to draw reader's attention to the versatility of the approach that can, in principle, be applied to other more geometrically complicated bodies where Gauss's law does not lead to simple solutions. Concurrently, this work also provides instructional approaches that intertwine content-specific and pedagogical viewpoints that can be useful to all undergraduate students and teachers who wish to enhance their command of the subject.

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I. INTRODUCTION

The Fourier transform method is one of the most useful tools to solve a large variety of problems in mathematics, physics and engineering. Its usefulness has been extended to many fields such as electromagnetism¹, sensor technologies² and quantum mechanics³⁻⁵. Undergraduate students in physics get a first glimpse of it when they take mathematical courses and, typically, see it as a useful method to solve differential equations⁶. Despite its great efficacy and wide applications, the Fourier transform method is rarely mentioned or used in typical calculus-based physics textbooks. We certainly agree that it is not a good idea to introduce this method way too early in a physics curriculum. However, by the same token, we do not see a strong reason why such a method is not introduced at least as a sole case study in advanced calculus-based physics courses that deal with electromagnetism phenomena⁷⁻¹². At this juncture of their academic degree program many undergraduate physics students have already studied the Fourier transform method in their mathematics courses.

In our opinion, this is a shortcoming that can be and should be addressed in calculus-based physics textbooks dealing with concepts of electromagnetism. Such textbooks already have very good case studies where at least one of them can be used to illustrate the formalism and implementation of the Fourier transform method in this context. After browsing the widely available literature, we identified one particular familiar problem covered virtually in all calculus-based physics textbooks dealing

with electromagnetism that can serve as a case study to illustrate the use of the Fourier transform method. This is the problem of a solid sphere with uniform volume charge density where the calculation of its electrostatic field, electrostatic potential and electrostatic self-energy is well known. Symmetry arguments enable Gauss's law application and from there one easily obtains the result for the electrostatic field at any arbitrary point in space. Obviously, the rest of the quantities, namely, the electrostatic potential and self-energy follow from basic formulas of electromagnetism.

Therefore, in this work, we reconsider the problem of a solid sphere with uniform volume charge density and explain pedagogically all the steps required to calculate its electrostatic self-energy by using Fourier transform techniques. We choose to use the Fourier transform method for the calculation of the self-energy to stress out the point that the Fourier method is not only useful to solve differential equations, but it also represents a very versatile tool to calculate complicated multi-variable integrals. In particular, we show for the case of a solid sphere with uniform volume charge density, how a 6-dimensional integral representing its self-energy can be reduced to a much more amenable form. The approach is very general and can be applied to more complicated bodies where Gauss's law does not help. Overall, the focus of our work is not to draw reader's attention to the difficulty of the problem solved. On the contrary, our main goal is to draw reader's attention to the methodology used and to the versatility of the approach that can, in principle, be applied to other charged bodies with a geometry different from that of a solid sphere¹³⁻¹⁷. In addition, use of

different methods to solve a given problem, many times leads to interesting insights or prospects to open a new discussion. For example, the Fourier method treatment shown in this work provides a good refresher of integration techniques as well as the opportunity to talk to undergraduate students about a class of special functions known as spherical Bessel functions if one chooses to do so.

The paper is organized as follows: In Section II we provide a quick discussion of known results for the case of a solid sphere with uniform volume charge density. In Section III we explain the Fourier method formalism and provide an easy to follow implementation of the method to solve the problem at hand. In Section IV we discuss the usefulness of the method from a pedagogical point of view and provide some concluding remarks.

II. SOLID SPHERE WITH UNIFORM VOLUME CHARGE DENSITY

Let's now consider the case study of a solid sphere with uniform volume charge density. We assume that the solid sphere has radius, R and contains a total positive charge, Q that is spread uniformly over its volume. The result is a constant uniform volume charge density:

$$\rho = \frac{3Q}{4\pi R^3} . \quad (1)$$

For convenience, one chooses a spherical system of coordinates with origin at the center of the solid sphere. Gauss's law allows one to obtain quite easily the electrostatic field of this body given its high symmetry. The result is well known. The electrostatic field vector created by a uniformly charged solid sphere has only a radial component with magnitude, $E(r) = |\vec{E}(\vec{r})|$ written as:

$$E(r) = \begin{cases} \frac{k_e Q}{R^3} r & ; \quad 0 \leq r < R , \\ \frac{k_e Q}{r^2} & ; \quad R \leq r < \infty , \end{cases} \quad (2)$$

where $r = |\vec{r}| \geq 0$ is the radial distance and $k_e = 1/(4\pi\epsilon_0)$ is Coulomb's electric constant. The calculation of the electrostatic potential, $V(r)$ follows directly from $E(r)$. The process is well explained in virtually all calculus-based physics textbooks. Basically, the starting point is the result:

$$E(r) = -\frac{dV(r)}{dr} \quad ; \quad dV(r) = -E(r) dr . \quad (3)$$

To obtain $V(r)$ one has to integrate over $E(r)$ and enforce the continuity of $V(r)$ all over the space for $0 \leq r < \infty$. The final expression is:

$$V(r) = \begin{cases} \frac{k_e Q}{2R} \left(3 - \frac{r^2}{R^2} \right) & ; \quad 0 \leq r < R , \\ \frac{k_e Q}{r} & ; \quad R \leq r < \infty . \end{cases} \quad (4)$$

The total electrostatic energy (self-energy) of a three-dimensional (3D) body with continuous but non-uniform charge distribution can be generally written as:

$$U = \frac{k_e}{2} \int_{\Omega} d^3r \int_{\Omega} d^3r' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} , \quad (5)$$

where $\rho(\vec{r})$ is the volume charge density at the specified location, Ω is a given volume domain containing the charge, $\rho(\vec{r}) d^3r$ and $\rho(\vec{r}') d^3r'$ represent elementary charges located around points $\vec{r}$ and $\vec{r}'$, respectively, d^3r and d^3r' represent elementary volumes and $\vec{r}$ and $\vec{r}'$ are 3D position vectors. As far as notation, the elementary volume in the literature is commonly written in various forms as $d^3r \equiv d^3\vec{r} \equiv \vec{dr} = dx dy dz = r^2 \sin(\theta) d\varphi d\theta dr$, where the last two equalities are in Cartesian and spherical coordinates, respectively.

Having all the results at disposal allows one to calculate the electrostatic self-energy of the uniformly charged solid sphere by various ways. For example, if one wants to calculate U from the electrostatic potential, the following expression that applies to a constant uniform volume charge density leads to:

$$U = \frac{\rho}{2} \int_{\Omega} d^3r V(r) = \frac{3}{5} \frac{k_e Q^2}{R} . \quad (6)$$

The same result as in Eq.(6) will be obtained if one calculates U from the electrostatic field:

$$U = \frac{\epsilon_0}{2} \int_{All Space} d^3r |\vec{E}(\vec{r})|^2 , \quad (7)$$

where now the integral is over all space (not confined to the volume region, Ω).

At this juncture, we may want to pause for a while and point out a crucial point. The reason why this approach works hinges solely on the fact that Gauss's law makes it very easy to calculate the electrostatic field (and from there the electrostatic potential). Had the body under consideration been a uniformly charged solid ellipsoid (but not a solid sphere) or a uniformly charged disk, then there is no way to solve this problem via this approach since Gauss's law will not tell us much about the electrostatic field. For all these situations, different methods should be applied where, in our opinion, the Fourier transform method stands out for its elegance and generality.

III. FOURIER TRANSFORM METHOD SOLUTION

Let's now change the formalism a little bit and rewrite the expression for the electrostatic self-energy as:

$$U = \frac{k_e \rho^2}{2} \int_{\Omega} d^3r_1 \int_{\Omega} d^3r_2 \frac{1}{|\vec{r}_1 - \vec{r}_2|} , \quad (8)$$

where ρ represents a uniform volume charge density and Ω is the region containing the charge which, in 3D spherical coordinates, is given as:

$$\Omega : \left\{ 0 \leq r \leq R ; 0 \leq \theta \leq \pi ; 0 \leq \varphi < 2\pi \right\} . \quad (9)$$

The slight change of notation in Eq.(8) relative to the original definition in Eq.(5) concerns the introduction of two new dummy variables denoted $\vec{r}_1$ and $\vec{r}_2$ and not $\vec{r}$ and $\vec{r}'$ as before. Note that we also used the fact that the volume charge density is uniform when we wrote the expression for U in Eq.(8) (in other words, ρ was moved out of the integral sign since it is a constant).

The quantity in Eq.(8) involves a complicated 6-dimensional integral. This is precisely the moment we were waiting for. The Fourier transform method will allow us to calculate the above 6-dimensional integral in a straightforward and elegant way. Furthermore, the approach is general and not specific to this particular case study.

To start with, let's define the pair of 3D Fourier transforms as:

$$\begin{cases} F(\vec{k}) = \int d^3r \exp(i \vec{k} \cdot \vec{r}) f(\vec{r}) , \\ f(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \exp(-i \vec{k} \cdot \vec{r}) F(\vec{k}) , \end{cases} \quad (10)$$

$$U = \frac{k_e \rho^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} \int_{\Omega} d^3r_1 \exp(-i \vec{k} \cdot \vec{r}_1) \int_{\Omega} d^3r_2 \exp(+i \vec{k} \cdot \vec{r}_2) , \quad (13)$$

where, for simplicity, we denoted $k = |\vec{k}| \geq 0$.

At this juncture, the first important remark is to note that the Fourier transform method approach is general up to this point and does not depend on the specific geometry of the body under consideration. The second important remark is to note that the approach allows one to simplify a complicated 6-dimensional integral (over 3D variables $\vec{r}_1$ and $\vec{r}_2$) into much simpler 3D integrals (albeit, with an added number of them). From now on the calculations are straightforward. Note the appearance in Eq.(13) of the following integral:

$$\int_{\Omega} d^3r \exp(\pm i \vec{k} \cdot \vec{r}) . \quad (14)$$

This integral is calculated in Appendix A and the result is:

$$\int_{\Omega} d^3r \exp(\pm i \vec{k} \cdot \vec{r}) = 4\pi R^3 \left[\frac{\sin(kR) - (kR) \cos(kR)}{(kR)^3} \right] . \quad (15)$$

With help from Eq.(15), one can write the expression in

where $F(\vec{k})$ is the 3D Fourier transform of $f(\vec{r})$, whereby $\vec{k}$ and $\vec{r}$ are 3D vectors, $i = \sqrt{-1}$ is the imaginary unit and the integration extends over all space. Note that we are explicitly using the dot product notation for the scalar product of any two vectors. We leave it as an exercise to the reader to verify that the 3D Fourier transform of function $f(\vec{r}) = 1/|\vec{r}|$ is function $F(\vec{k}) = 4\pi/|\vec{k}|^2$. In other words, one has:

$$\frac{4\pi}{|\vec{k}|^2} = \int d^3r \exp(i \vec{k} \cdot \vec{r}) \frac{1}{|\vec{r}|} . \quad (11)$$

This means that, based on the second formula in Eq.(10), one can write:

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{(2\pi)^3} \int d^3k \exp[-i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)] \frac{4\pi}{|\vec{k}|^2} . \quad (12)$$

By substituting the result from Eq.(12) into Eq.(8) and reordering the integrals one obtains:

Eq.(13) as:

$$U = \frac{k_e \rho^2}{2} (4\pi R^3)^2 \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} \left[\frac{\sin(kR) - (kR) \cos(kR)}{(kR)^3} \right]^2 . \quad (16)$$

It is easy to see from Eq.(1) that $\rho 4\pi R^3 = 3Q$. This allows us to write:

$$U = \frac{k_e}{2} (3Q)^2 \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} \left[\frac{\sin(kR) - (kR) \cos(kR)}{(kR)^3} \right]^2 . \quad (17)$$

The integral in Eq.(17) is elementary. One writes $d^3k = 4\pi k^2 dk$ and by introducing a dummy variable $x = kR$ one has:

$$U = \frac{9}{\pi} \frac{k_e Q^2}{R} \int_0^\infty dx [f(x)]^2 , \quad (18)$$

where $f(x)$ represents the following auxiliary function:

$$f(x) = \frac{\sin(x) - x \cos(x)}{x^3} . \quad (19)$$

We leave it to the reader to verify that $\int_0^\infty dx [f(x)]^2 = \pi/15$. As a result, the electrostatic self-energy value in

Eq.(18) becomes :

$$U = \frac{3}{5} \frac{k_e Q^2}{R} . \quad (20)$$

This is the expected result for the electrostatic self-energy of a solid sphere with uniform volume charge density. As already stated, the Fourier transform method is useful not because it can solve this specific case study, but because it serves as a powerful tool to solve much more complicated problems in which charge distributions have neither spherical, nor cylindrical symmetry. For instance, the electrostatic self-energy of a uniformly charged elliptical plate¹⁸ has been calculated by adopting the same procedure bearing in mind the only straightforward modifications that come from a lower dimensionality. Therefore, we take the opportunity to reinforce the point that the same Fourier transform formalism sketched out here can be adopted to obtain the self-energy of other more complicated bodies such as uniformly charged solid ellipsoids, uniformly charged elliptical disks and likewise.

IV. DISCUSSION AND CONCLUSIONS

In this work we show how to use the Fourier transform method as a powerful tool to solve problems that arise in a typical calculus-based physics course that covers topics of electromagnetism. Such a method is well known to undergraduate students from various courses in mathematics, but its implementation is lacking in physics textbooks at this level. The idea is not to use the method to solve a particularly challenging problem, but to illustrate its implementation through a well thought example. With this in mind, we zeroed in on the case study of the electrostatic self-energy of a solid sphere with uniform volume charge density which fits well this mindset. This model is very common and appears in virtually all calculus-based physics books in electromagnetism.

As illustrated in a detailed pedagogical way, the Fourier transform method is general and, in principle, applies to any arbitrary shape for a given charged body. Another very important result to have in mind is that the Fourier transform approach always allows one to simplify the rather challenging 6-dimensional self-energy integral (integration over 3D variables $\vec{r}_1$ and $\vec{r}_2$ within the volume of the body) into simpler 3D integral pieces (integration within the volume of the same body). This is a considerable reduction of the difficulty of the problem given that 3D integrals over a typical regular body generally lead to analytical results. On the other hand, this is also a useful simplification for numerical calculations^{19–22} of the electrostatic self-energy of charged bodies with irregular shape since a numerical integration with three variables can be done faster and more accurately than its counterpart with six variables.

Another important message to convey is the new perspective gained when solving problems using different

methods and approaches. For instance, the current application of the Fourier transform method to this specific problem allows one to cover nicely various integration techniques for a spherical system of coordinates. This is always a good refresher for an audience of undergraduate students when it comes to solving physics problems that involve integrals. On top of that, one may use the presented opportunity to discuss even more challenging mathematical topics if the audience is more mathematically inclined than a typical one. For example, one may use this occasion to expose an audience of undergraduate students to certain special functions that routinely are not mentioned at this level. For instance, a keen observer may have immediately noticed that the expression in Eq.(A8) may be written in a more compact form as:

$$I(\vec{k}) = 4\pi R^3 \frac{j_1(kR)}{(kR)} , \quad (21)$$

where

$$j_1(x) = \frac{\sin x - x \cos x}{x^2} , \quad (22)$$

is known as a spherical Bessel function of the first kind. This is a good moment to impress the audience with the importance of many types of Bessel functions of various kinds²³ that show up so often in many science, technology, engineering and mathematics disciplines. In a nutshell, this work exposes the appropriate targeted audience to a very useful tool that can be used to enrich teaching and learning of calculus-based physics at undergraduate level. In this sense, we believe that the results reported have clear pedagogical value and may be of interest to the specialized, as well as to the broad audiences of students and teachers.

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APPENDIX A: INTEGRAL $\int_{\Omega} d^3r \exp(\pm i\vec{k} \cdot \vec{r})$ OVER A 3D SOLID SPHERE DOMAIN

Let's consider the following 3D integral (we consider first the case with a positive sign):

$$I(\vec{k}) = \int_{\Omega} d^3r \exp(+i\vec{k} \cdot \vec{r}) , \quad (A1)$$

where

$$\Omega : \{0 \leq r \leq R ; 0 \leq \theta \leq \pi ; 0 \leq \varphi < 2\pi\} , \quad (A2)$$

is the integration domain in spherical coordinates. The solid sphere has a radius R , while $\vec{k}$, $\vec{r}$ are 3D vectors and $i = \sqrt{-1}$ is the imaginary unit number. We adopt a 3D spherical system of coordinates where $\vec{k}$ is imagined to lie along the positive z -direction. With this choice of the system of coordinates, the quantity in Eq.(A1) is written as:

$$I(\vec{k}) = \int_0^R dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \exp(+i k r \cos\theta) , \quad (\text{A3})$$

where $k = |\vec{k}| \geq 0$ and $r = |\vec{r}| \geq 0$. At this point, we introduce a new variable, $t = \cos\theta$ and rewrite the term in Eq.(A3) as:

$$I(\vec{k}) = (2\pi) \int_0^R dr r^2 \int_{-1}^{+1} dt \exp(+i k r t) . \quad (\text{A4})$$

One can easily prove that the integration over variable t leads to the following expression:

$$I(\vec{k}) = (4\pi) \int_0^R dr r^2 \frac{\sin(kr)}{(kr)} . \quad (\text{A5})$$

By changing to a new dummy variable, $x = kr$, one rewrites Eq.(A5) as:

$$I(\vec{k}) = \frac{4\pi}{k^3} \int_0^{kR} dx x \sin(x) . \quad (\text{A6})$$

We now use of the standard integration formula:

$$\int dx x \sin(x) = \sin(x) - x \cos(x) . \quad (\text{A7})$$

This leads to a final result that we prefer to write as:

$$I(\vec{k}) = 4\pi R^3 \left[\frac{\sin(kR) - (kR) \cos(kR)}{(kR)^3} \right] . \quad (\text{A8})$$

If one recalls that $\sin x \approx x - x^3/3! + 0(x^5)$ and $\cos x \approx 1 - x^2/2! + 0(x^4)$ for $x \approx 0$, then it is easy to verify, starting from Eq.(A8), that:

$$\lim_{\vec{k} \rightarrow 0} I(\vec{k}) = \frac{4\pi}{3} R^3 . \quad (\text{A9})$$

Note that the quantity $I(\vec{k})$ in Eq.(A8) is real. This means that by complex conjugating both sides of Eq.(A1) one can easily see that:

$$\int_{\Omega} d^3r \exp(-i \vec{k} \cdot \vec{r}) = I(\vec{k})^* = I(\vec{k}) , \quad (\text{A10})$$

where the symbol of the asterisk (*) means complex conjugation. In a nutshell, we proved that:

$$\int_{\Omega} d^3r \exp(\pm i \vec{k} \cdot \vec{r}) = I(\vec{k}) = 4\pi R^3 \left[\frac{\sin(kR) - (kR) \cos(kR)}{(kR)^3} \right] . \quad (\text{A11})$$

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