Circular motion of a charged particle in a uniform constant magnetic field revisited

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The classical motion of a charged particle in a uniform constant magnetic field is well-studied in physics courses. The resulting motion is circular if the magnetic field is perpendicular to the initial velocity of the particle. The equations of motion involve a set of coupled differential equations whose solutions are typically not provided in an undergraduate calculus-based physics course. While this is understandable, it is quite surprising that even the expressions for the coordinates of the center of the circular orbit in terms of initial conditions are generally missing. Even more specialized textbooks focused on electromagnetism and analytical mechanics lack the desired details. In this work, we treat this problem in such a pedagogical way that we believe we are able to address all these shortcomings. To this effect, we first lay out all the details involved in the standard solution of the problem. We also provide the expression for the center of the circular orbit in terms of initial position, initial velocity and cyclotron angular frequency. Secondly, we revisit the problem by introducing another solution method that uses complex coordinates. This second mathematical approach is elegant and allows one to study the motion via a single second order linear homogeneous differential equation with constant complex coefficients rather than two coupled differential equations that are not of the separated type. The work draws attention to certain pedagogical aspects that require more attention for a better understanding of this problem.

Keywords: Magnetic field; Cyclotron motion; Charged particle; Equations of motion; Differential equations.

I. INTRODUCTION

A uniform constant magnetic field can cause charged particles to move in circular or helical paths. Examples of this behavior are plenty such as the circular paths of protons in particle accelerators, the helical motion of charged particles in Earth's atmosphere when they encounter Earth's magnetic field and so on. Motion in curved paths for charged particles in a magnetic field is the basis of many important phenomena in science and technology. The quantum counterpart of this problem leads to the physics of Landau states and many other important quantum phenomena [1–13]. Understanding the classical motion of a charged particle in a uniform constant magnetic field is relatively straightforward from the point of view of classical physics. The basis to understand it lies in the nature of the magnetic force that enters the Lorentz law expression. The magnetic force has a direction perpendicular to both the velocity and the magnetic field and is proportional to the magnitude of charge. Since the magnetic force is always perpendicular to velocity, it does no work on the charged particle. As a result, the particle's kinetic energy and speed (magnitude of velocity) remain constant. The direction of motion is affected, but not the speed. This behavior is typical of uniform circular motion. The simplest case occurs when a charged particle has a velocity that is perpendicular to a uniform constant magnetic field. In this case the magnetic force represents the centripetal force and the charged particle follows a circular trajectory with a given radius and a specific center of rotation. This is the description that is routinely found in all undergraduate algebra-based and/or calculus-based physics textbooks [14–17].

The problem of the classical motion of a charged particle in a uniform constant magnetic field is also covered in more advanced electromagnetism/electrodynamics textbooks [18–21]. The provided explanation of the phenomenon is more or less along the same qualitative lines as already mentioned. As one knows the Newtonian equations of motion for this case lead to a set of coupled differential equations. The details of the solution are, presumably, deemed a little bit too technical and are not provided in a typical undergraduate calculus-based physics textbook [14–17]. This course of action can be justified if one argues that simplicity is a paramount objective. However, it is quite surprising to us, to observe that some other major details are missing from the general solution of this problem. For instance, the expressions for the location of the center of the circle of rotation in terms of the initial position/velocity and other parameters of the system (angular frequency) are never given. The exact mathematical expression for such a quantity (that is not very complicated) is remarkably missing from all the literature surveyed earlier [14–21]. At this stage, any student has the right to ask the following question: Assuming that a charged particle is moving in a circle, what is the formula for the location of the center of this circle? Obviously, this important parameter is determined by the initial conditions for position/velocity of the charged particle in conjunction with other factors and, at least, must be stated as an important result. Even other more specialized textbooks that are focused on analytical mechanics and classical dynamics and have a higher dose of mathematics do not provide a good answer to this question in sufficient details [22–24].

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Therefore, the purpose of this work is to address these shortcomings in a pedagogical manner that leads to a treatment that is attractive to students and researchers alike. More specifically speaking, this work has two objectives. The first objective is to provide all the details of the standard mathematical method that is commonly used to solve the problem of the circular motion of a charged particle in a uniform constant magnetic field. To this effect, we will also derive the expression for the center of the circular orbit in terms of initial position, initial velocity and cyclotron angular frequency. This step-by-step approach will stress key pedagogical facts that should be part of any treatment of this problem. The second objective of the work is to revisit the problem from a different mathematical perspective. To this effect, we will explain a different solution method of the problem that uses complex variables [25]. This mathematical approach is elegant and allows one to study the resulting two-dimensional (2D) motion of a charged particle in a perpendicular magnetic field via a single second order linear homogeneous differential equation rather than two coupled differential equations that are not of the separated type.

The article is organized as follows. In Section II we introduce the model and explain the details of the standard solution method. In Section III we describe the details of a different solution method that uses complex variables. In Section IV we briefly summarize the key aspects of the work from a pedagogical point of view.

II. MODEL AND STANDARD SOLUTION

Let us consider a particle of mass m, positive charge $q(>0)$ moving in a uniform constant magnetic field. Let the z-axis be chosen in the direction of the magnetic field:

$$\vec{B} = (0, 0, B) \ , \tag{1}$$

where $B > 0$ is the magnitude of the magnetic field. The initial position coordinates and initial velocity components are chosen, respectively, as:

$$x(t=0) = x_0 \ ; \ y(t=0) = y_0 \ ; \ z(t=0) = z_0 = 0 \ , \tag{2}$$

and

$$v_x(t=0) = v_{0x} \ ; \ v_y(t=0) = v_{0y} \ ; \ v_z(t=0) = v_{0z} = 0 \ . \tag{3}$$

This means that the initial position vector and initial velocity vector are in the $x-y$ plane while the magnetic field is perpendicular to them in the positive z-direction. If $\vec{v}$ is the velocity, the magnetic force $\vec{F} = q\,\vec{v}\times\vec{B}$ will cause the particle to move in a circle in the $x-y$ plane as schematically shown in Fig. 1. In order to find the time-dependent position and time-dependent velocity of the particle at any moment of time we need to solve the Newtonian equation:

$$m\,\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B} \ . \tag{4}$$

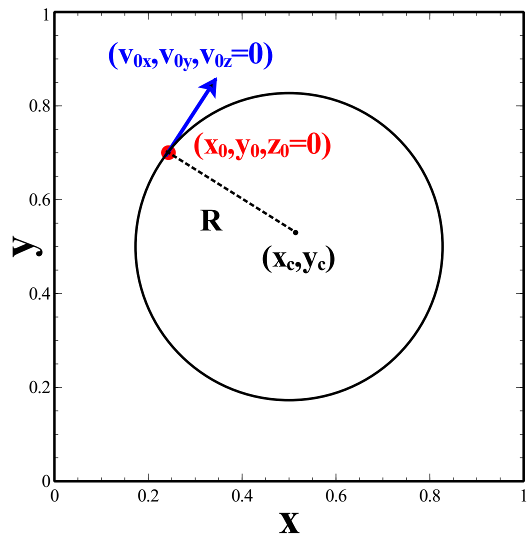

FIG. 1. A positively charged particle is moving in a uniform constant magnetic field. The initial position vector and initial velocity vector are in the $x-y$ plane while the magnetic field is perpendicular to them. The direction of the magnetic field is out of the page in the positive z-direction (not shown). When the velocity of charged particle is perpendicular to a uniform constant magnetic field, the particle moves in a circular path in a plane perpendicular to the magnetic field.

It is easy to check that, based on the conditions in Eq.(1), Eq.(2), Eq.(3) and Eq.(4), there is no dynamics in the z-direction. This means that $v_z(t) = v_{0z} = 0$ and $z(t) = z_0 = 0$.

Therefore, the quantities to calculate are the velocity components on the $x-y$ plane, $v_x(t)$, $v_y(t)$ and the position components on the $x-y$ plane, $x(t)$ and $y(t)$. After equating like vector components for the x and y directions in Eq.(4), one obtains the following equations of motion:

$$\begin{cases} \dot{v}_x = +\omega_c\,v_y \\ \dot{v}_y = -\omega_c\,v_x \ , \end{cases} \tag{5}$$

where

$$\omega_c = \frac{q\,B}{m} > 0 \ , \tag{6}$$

is the so-called cyclotron angular frequency and, in a short-hand notation, $\dot{v}_{x,y} = dv_{x,y}/dt$ denotes a first derivative with respect to time.

Typical calculus-based undergraduate physics textbooks do not provide the details of the solution for the differential equations in Eq.(5). More advanced textbooks such as those focused on classical dynamics [23] and/or mechanics [24] provide some of the steps. Apart differences of notation, the common solution method is to uncouple the coupled simultaneous differential equations in Eq.(5) by differentiating one of them, for instance, the first one in Eq.(5) which becomes: $\ddot{v}_x = +\omega_c\,\dot{v}_y$ and use the other one ($\dot{v}_y = -\omega_c\,v_x$) to eliminate one of the variables (in this case, v_y). This process leads to the following differential equation:

$$\ddot{v}_x = -\omega_c^2\,v_x\ . \tag{7}$$

This equation has the same form as the differential equation for a one-dimensional (1D) harmonic oscillator with angular frequency ω_c. It is a second order linear homogeneous differential equation with constant coefficients with the general solution that reads:

$$v_x(t) = c_1\,\cos(\omega_c\,t) + c_2\,\sin(\omega_c\,t)\ , \tag{8}$$

where c_1 and c_2 are arbitrary constants (for now). Let's take the expression for $v_x(t)$ in Eq.(8) and use it in the second equation in Eq.(5):

$$\dot{v}_y = -\omega_c\,v_x(t) = -\omega_c\,c_1\,\cos(\omega_c\,t) - \omega_c\,c_2\,\sin(\omega_c\,t)\ . \tag{9}$$

From here one can calculate $v_y(t)$ by integration:

$$v_y(t) = -c_1\,\sin(\omega_c\,t) + c_2\,\cos(\omega_c\,t) + c_3\ , \tag{10}$$

where c_3 is a new arbitratry integration constant that comes from the integration and, technically, should be included. However, the typical approach (without much of an explanation) is not to incorporate such a constant hinting to the result that:

$$c_3 = 0\ . \tag{11}$$

The reason why so draws a very legitimate question.

We feel that a careful explanation of this point is fully warranted. After all, this omission is related to a very delicate mathematical step that is tacitly implied in the literature, but is not adequately explained. Therefore, we believe that this subtle mathematical point is worth discussing in detail. As made clear throughout this approach, the results from Eq.(8) and Eq.(10) were obtained by initially differentiating the first equation in Eq.(5). Differentiating an equation may introduce new solutions that do not satisfy the original equation. Consider the very simple equation $x = 1$. Differentiating, we get $\dot{x} = 0$. Its solution is $x = c$ where c is any arbitrary constant. Note that only for one particular value of the constant c will this equation satisfy the original equation. This means that one should make sure to verify that the two solutions that were obtained by this approach satisfy the original equations in Eq.(5) by going back to them. Only by doing so, one sees that $v_x(t)$ from Eq.(8) and $v_y(t) = -c_1\,\sin(\omega_c\,t) + c_2\,\cos(\omega_c\,t) + c_3$ from Eq.(10) satisfy simultaneously the original differential equations in Eq.(5) only when the condition $c_3 = 0$ is imposed. This is the reason why the common approach in many cases is to omit the integration constant c_3 and immediately write $v_y(t)$ as:

$$v_y(t) = -c_1\,\sin(\omega_c\,t) + c_2\,\cos(\omega_c\,t)\ . \tag{12}$$

However, the details that lead to such a conclusion are rarely mentioned. Therefore, we believe that an audience of undergraduate students and physics teachers need to be aware of all these mathematical subtleties in order to have a full mastery of this solution method in case one attempts to generalize it to other problems.

The specific value of the two arbitrary constants c_1 and c_2 is determined by the initial conditions in Eq.(3) leading to:

$$\begin{cases} v_x(t) = +v_{0x}\,\cos(\omega_c\,t) + v_{0y}\,\sin(\omega_c\,t) \\ v_y(t) = -v_{0x}\,\sin(\omega_c\,t) + v_{0y}\,\cos(\omega_c\,t)\ . \end{cases} \tag{13}$$

Integrating $v_{x,y}(t)$ leads to the equations for $x(t)$ and $y(t)$ written as:

$$\begin{cases} x(t) = \int v_x(t)\,dt + x_c \\ y(t) = \int v_y(t)\,dt + y_c\ , \end{cases} \tag{14}$$

where x_c and y_c are two arbitrary constants that come from the integration. The notation hints to the expectation that the constants x_c and y_c will be shown to represent the coordinates of the center of the circle of rotation of the charged particle. After completing the integration, the result reads:

$$\begin{cases} x(t) = x_c + \frac{v_{0x}}{\omega_c}\,\sin(\omega_c\,t) - \frac{v_{0y}}{\omega_c}\,\cos(\omega_c\,t) \\ y(t) = y_c + \frac{v_{0x}}{\omega_c}\,\cos(\omega_c\,t) + \frac{v_{0y}}{\omega_c}\,\sin(\omega_c\,t)\ . \end{cases} \tag{15}$$

The specific values of the two arbitrary constants x_c and y_c are determined from the initial position conditions in Eq.(2) with the final result:

$$\begin{cases} x_c = x_0 + \frac{v_{0y}}{\omega_c} \\ y_c = y_0 - \frac{v_{0x}}{\omega_c}\ . \end{cases} \tag{16}$$

Let us now explicitly show that, indeed, x_c and y_c represent the coordinates of the center of the circle of rotation of the charged particle. To that effect, let us rewrite Eq.(15) as:

$$\begin{cases} x(t) - x_c = \frac{v_{0x}}{\omega_c}\,\sin(\omega_c\,t) - \frac{v_{0y}}{\omega_c}\,\cos(\omega_c\,t) \\ y(t) - y_c = \frac{v_{0x}}{\omega_c}\,\cos(\omega_c\,t) + \frac{v_{0y}}{\omega_c}\,\sin(\omega_c\,t)\ . \end{cases} \tag{17}$$

It is a trivial exercise to check that:

$$[x(t) - x_c]^2 + [y(t) - y_c]^2 = R^2\ , \tag{18}$$

where

$$R^2 = \frac{v_{0x}^2 + v_{0y}^2}{\omega_c^2}\ . \tag{19}$$

Obviously, Eq.(18) represents a circle with center located at (x_c, y_c) where the coordinate values, x_c and y_c are specified by Eq.(16). The radius of such a circle is $R =$

$\sqrt{(v_{0x}^2 + v_{0y}^2)}/\omega_c$ as specified from Eq.(19). To conclude, the charged particle's trajectory in the 2D plane normal to $\vec{B}$ is, therefore, a circle with center at (x_c, y_c) and radius R. As a final remark, one can write the position coordinates in Eq.(15) in much more compact form as:

$$\begin{cases} x(t) = x_c - \frac{v_y(t)}{\omega_c} \\ \\ y(t) = y_c + \frac{v_x(t)}{\omega_c} \ , \end{cases} \tag{20}$$

where (x_c, y_c) are given from Eq.(16) and $(v_x(t), v_y(t))$ are given from Eq.(13).

III. DIFFERENT METHOD WITH COMPLEX VARIABLES

Let us now revisit the solution of the coupled simultaneous differential equations in Eq.(5) from a different perspective. Rather than differentiating one of the equations as we did earlier, let us multiply both sides of the second equation in Eq.(5) by the imaginary unit, $i = \sqrt{-1}$ and rewrite the two quantities as:

$$\begin{cases} \dot{v}_x = -i\,\omega_c\,(i\,v_y) \\ \\ i\,\dot{v}_y = -i\,\omega_c\,v_x \ . \end{cases} \tag{21}$$

The idea is to solve exactly the system of coupled differential equations in Eq.(21) by using complex variables. Adding side by side the expressions in Eq.(21) leads to:

$$\dot{v}_x + i\,\dot{v}_y = -i\,\omega_c\,(v_x + i\,v_y) \ . \tag{22}$$

At this point, we define a complex 2D position variable as:

$$z = x + i\,y \ . \tag{23}$$

We believe that, at this juncture, the readers are well aware not to confuse the 2D complex variable z defined in Eq.(23) (or its initial value z_0 that will be defined later, etc.) with the position coordinate in the z direction that was briefly mentioned earlier in Eq.(2). With this understanding, the 2D complex velocity reads as: $v = v_x + i\,v_y = \dot{z}$ and, similarly, the 2D complex acceleration can be written as: $\dot{v} = \dot{v}_x + i\,\dot{v}_y = \ddot{z}$. Therefore, one can rewrite the result in Eq.(22) as:

$$\ddot{z} + i\,\omega_c\,\dot{z} = 0 \ , \tag{24}$$

where the initial time conditions for the 2D complex position and 2D complex velocity are, respectively:

$$z(t=0) = z_0 = x_0 + i\,y_0 \ , \tag{25}$$

and

$$v(t=0) = v_0 = v_{0x} + i\,v_{0y} \ . \tag{26}$$

This means that, without any need to take an extra differentiation, we have managed to express the original system of two coupled differential equations as a single differential equation in complex notation. The result in Eq.(24) represents a second order linear homogeneous differential equation with constant complex coefficients. If the constants are complex numbers [26], it is still possible to find solutions of the form $z(t) = \exp{(r\,t)}$ where r satisfies the root equation which in this case is:

$$r^2 + i\,\omega_c\,r = 0 \ . \tag{27}$$

The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -i\,\omega_c$. Therefore, the general solution for the 2D complex position is:

$$z(t) = c_1 + c_2\,\exp{(-i\,\omega_c\,t)} \ , \tag{28}$$

where c_1 and c_2 are two arbitrary complex constants. The 2D complex velocity is obtained by differentiating $z(t)$ in Eq.(28) with respect to time:

$$v(t) = -i\,\omega_c\,c_2\,\exp{(-i\,\omega_c\,t)} \ . \tag{29}$$

The two complex constants, c_1 and c_2 are determined from the initial conditions, $z(t=0) = z_0$ and $v(t=0) = v_0$ which lead to:

$$c_1 + c_2 = z_0 \ ; \ -i\,\omega_c\,c_2 = v_0 \ . \tag{30}$$

As a result the two complex constants c_1 and c_2 are:

$$c_1 = z_0 - i\,\frac{v_0}{\omega_c} \ ; \ c_2 = i\,\frac{v_0}{\omega_c} \ . \tag{31}$$

One can write the complex constants, c_1 and c_2 in terms of their real and imaginay parts:

$$c_1 = c_1' + i\,c_1'' \ ; \ c_2 = c_2' + i\,c_2'' \ . \tag{32}$$

This allows us to obtain:

$$c_1' = x_0 + \frac{v_{0y}}{\omega_c} \ ; \ c_1'' = y_0 - \frac{v_{0x}}{\omega_c} \ , \tag{33}$$

and

$$c_2' = -\frac{v_{0y}}{\omega_c} \ ; \ c_2'' = \frac{v_{0x}}{\omega_c} \ . \tag{34}$$

By separating the real and imaginary parts of Eq.(28), and taking into account that $\exp(-i\,\omega_c\,t) = \cos(\omega_c\,t) - i\,\sin(\omega_c\,t)$, one obtains the positions of the particle, $x(t)$ and $y(t)$ as a function of time:

$$\begin{cases} x(t) = c_1' + c_2''\sin{(\omega_c\,t)} + c_2'\cos{(\omega_c\,t)} \\ \\ y(t) = c_1'' + c_2''\cos{(\omega_c\,t)} - c_2'\sin{(\omega_c\,t)} \ . \end{cases} \tag{35}$$

By comparing the results from Eq.(33) to those in Eq.(16), one immediately identifies: $c_1' = x_c$ and $c_1'' = y_c$. A comparison of the expression from Eq.(35) with constants given from Eq.(33) and Eq.(34) to the expressions in Eq.(15) allows one to see that the two results are identical. One obtains velocities, $v_x(t)$ and $v_y(t)$ by differentiating the positions $x(t)$ and $y(t)$ with respect to time, respectively.

At this juncture, we want to point out that the crucial important result obtained by means of this method is the one for the 2D complex position $z(t)$ in Eq.(28). This result represents the solution to the second order linear homogeneous differential equation with constant complex coefficients in Eq.(24). The rest of the treatment involves straightforward algebraic manipulations and book-keeping which enables the reader to carefully check the correctness of the final results. Therefore, the beauty of this approach is in the elegance of solving the differential equations of motion in such a way that: (i) A system of two coupled simultaneous differential equations is reduced to a single one in complex coordinates; (ii) There is no need for any additional differentiation of any of the original differential equations and (iii) The solution of the resulting second order linear homogeneous differential equation with constant complex coefficients is elementary.

IV. CONCLUSIONS

The classical motion of a charged particle in a uniform constant magnetic field is covered in all physics courses at both the undergraduate and graduate level. The most interesting scenario that develops is that of circular motion which arises when the magnetic field is perpendicular to the initial velocity of the particle. The Newtonian equations of motion involve a set of coupled differential equations. The details of the solutions are typically not given in an undergraduate calculus-based physics course. While this approach keeps the level of mathematics simple, it is quite surprising to see that even the expressions for the coordinates of the center of the circular orbit generally are not provided. More specialized textbooks focused on electromagnetism, analytical mechanics or classical dynamics provide some extra steps when it comes to the solution of the mathematical problem. However, some key details and a few subtle explanations are missing from the treatment.

In this work we fill some of these voids in such a way that should be appealing to both students and physics teachers. Firstly, we provide all the key details for the conventional solution method widely used in the literature. This process smoothly leads to the expected equations of motion including the result for the location of the center of the circular orbit in terms of initial position, initial velocity and cyclotron angular frequency. Secondly, we revisit this problem by applying a different mathematical solution method that uses complex coordinates. In our view, while being a little bit more demanding (since it requires some basic knowledge of complex numbers), this approach is very appealing because of its elegance. Moreover, this method allows one to study the 2D motion of the charged particle via a single second order linear homogeneous differential equation with constant complex coefficients rather than the initial pair of coupled differential equations that are not of the separated type. In a nutshell, this work highlights some key pedagogical aspects of this problem that require more attention for a better understanding of the whole treatment. This means that, in principle, the results of this work should be welcomed by a broad audience of students, teachers and researchers working in various scientific disciplines.

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