

About approximations in some classic relativistic effects in a weak central symmetric gravitational field in undergraduate level

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Draft

In this paper some aspects concerning approximations which can be made in calculating two important effects resulting from General Relativity, namely the perihelion shift of planets moving around the Sun (Mercury) and bending of starlight observed on Earth grazing the apparent Sun's disc in its trajectory are discussed. The simplifications are presented in the paper and comparison is made between Njntonian, so called "eclectic" and GR approach. As a particular case of approximation a perihelion shift quantity between "eclectic" approach and correct GR could be found. Schwarzschild solution and Lagrangian approaches are throughout used in this paper. The discussion presented here seems to the author to be of pedagogical importance in teaching General Relativity in undergraduate level.

I. Introduction

It is well known that the so-called "classic" historic relativistic effects in General Relativity-GR, such as gravitational redshift of spectral lines, perihelion shift of planets and bending of light in the field of the Sun are "quite naturally" derived in many standard textbooks from the Schwarzschild metric and the use of geodesic equation: see for example^{2,4,8}. The Schwarzschild metric and corresponding geodesic equation are^{*}

$$ds^2 = (1 - r_s / r) c^2 dt^2 - ((1 - r_s / r))^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\alpha^2 \quad (1)$$

$$d^2 x^\mu / dk^2 + \Gamma^\mu_{\nu\lambda} (dx^\nu / dk) (dx^\lambda / dk) = 0 \quad (2)$$

In these formulae, we note the so-called Schwarzschild radius $r_s \equiv 2GM_\odot / c^2 \equiv 2\bar{M}$ with $\bar{M} \equiv GM_\odot / c^2$ - mass of the Sun in *geometrized units*- and a geodesic parameter k , properly chosen for deriving these effects and not only. Spatial coordinates are r, θ, α -respectively radial, polar and azimuthal coordinate.

The equations of the motion of massive particles or massless ones (photons) are derived from a Lagrangian below[†]:

$$\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (3),$$

where the dot is used to express the derivative of generalized coordinates (space time coordinates) in respect to the geodesic parameter and $g_{\mu\nu}(x)$ is the fundamental metric tensor.

It is well known¹ that Newton theory, in the framework of absolute space-time and Newtonian gravitational law can qualitatively yield a perihelion shift if, by side of the gravitational potential of the order of a/r , one adds a potential of the order of b/r^2 , or even

^{*} Throughout in this paper, we use the signature convention (+, -, -, -)

[†] We use symbol $\mathcal{L}$ for Lagrangian instead of L which is kept to present angular momentum symbol

c/r^3 , where a, b, c are constants. These parameters should be taken from a basic theory. Whereas a is well defined from the conventional Newtonian Theory, other constants can be taken from nowhere, as well as respective potentials of higher order, because we don't have an original basic theory-as far as two body case is concerned, but it has been shown that taking into account a perturbation of the orbit caused by another planet the term of the order γ/r^3 can be obtained⁵. On the other hand, it has been found that corrections to the inverse-square gravitational force law formally similar to that required by general relativity were suggested by Clairaut in the 18th century. By the way, in ref.¹¹ the calculation of correct perihelion shift is made using complex functions.

When teaching relativity to undergraduate students this particular aspect should, in our opinion, be emphasized, for obvious pedagogical reasons. We will demonstrate-next in our paper, that GR can quite naturally put in evidence such terms as well as respective constants, when going to the second approximation, in calculating perihelion shift in a weak gravitational field and flat three-dimensional space. We will show that in this framework one can obtain a shift which is (2/3)-d of the exact result obtained by full treatment of the problem in the context of Schwarzschild metric, made in standard textbooks of GR. In the latter case, we are in the realm of curved space-time and three-dimensional curved space.

In this paper, we discuss the possibility of calculation of some mentioned effects using appropriate approximations, in one hand and using some "eclectic" reasoning on the other. These calculations and this approach seem to the author to be illuminating and important from a pedagogical point of view at least.

II. Perihelion shift from an "eclectic" point of view

We show here that one can calculate a perihelion shift based on an "eclectic" reasoning, as we call it, combining together the Special Theory of Relativity and Newtonian gravitation. On these grounds, we are dealing with a flat three-dimensional space considered in an inertial frame with the Sun at the center of space coordinates, an inertial mass changing with its velocity and a Newtonian gravitational mass not changing during the motion. This is of course a "pure eclectic" reasoning, but still we proceed to derive the effect and compare its result with the correct Einsteinian result, which has been successfully confirmed by observation many times.

We start with defining the Lagrangian of the planet in spherical spatial coordinates with the Sun at its center which looks like this:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\dot{r}^2 + r^2 \dot{\alpha}^2}{c^2}} + \frac{GM_{\odot}m}{r} \quad (4),$$

with $\dot{r} = dr/dt, \dot{\alpha} = d\alpha/dt$, where m is the rest mass of the planet (in our case equal to Newtonian gravitational mass), $M_{\odot}$ is the mass of the Sun and r is the radial coordinate. In our case coordinate r is also the true distance of the planet from the center of the Sun and not simply a radial coordinate as it is in the Schwarzschild metric (1). Other coordinates are polar angle θ and azimuthal angle α . The time t is a true time of the inertial system of the Sun. The planet moves in this time and this inertial system.

In the case we neglect effects of second order relative to terms v/c , where v is the velocity of the planet, we have an approximate Lagrangian

$$\mathcal{L}^* = -mc^2 + \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\alpha}^2}{2} + \frac{GM_{\odot}m}{r} \quad (4')$$

Euler-Lagrange equation for azimuthal angle $\alpha(t)$ relating equation (4) is

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right)_{\alpha} = \left(\frac{\partial \mathcal{L}}{\partial \alpha} \right)_{\dot{\alpha}} \quad (5),$$

which yields the constant of the motion-the angular moment L

$$L = \gamma m r^2 \dot{\alpha} = \text{const.} \quad (5'),$$

where $\gamma \equiv \left(1 - \left(\dot{r}^2 + r^2 \dot{\alpha}^2 \right) / c^2 \right)^{-1/2}$ is the well-known relativistic factor of Special Relativity.

We chose as usual the plane of the orbit to be that for $\theta = \pi/2 = \text{const.}$

Radial equation is:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right)_r = \left(\frac{\partial \mathcal{L}}{\partial r} \right)_{\dot{r}} \quad (5'')$$

which from (4) yields:

$$\frac{d}{dt} \left(\gamma m \dot{r} \right) = \gamma m r^2 \dot{\alpha} + \frac{GM_{\square} m}{r^2} \quad (6)$$

Further, one proceeds as follows.

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{d\alpha} \frac{d\alpha}{dt} = \dot{\alpha} r' = \frac{L}{\gamma m r^2} r'; r' \equiv \frac{dr}{d\alpha} \quad (6')$$

Taking a new function $u(\alpha) \equiv 1/r(\alpha)$, as one usually makes when discussing Kepler problem, and after using (6,6'), we find the differential equation for the new function $u(\alpha)$:

$$-\frac{L^2}{\gamma m} u^2 u'' = L^2 u^3 - GM_{\square} \gamma m^2 u^2$$

or

$$u'' + u - GM_{\square} \gamma m^2 / L^2 = 0 \quad (7)$$

By adding together, the kinetic and the potential energy, as to have the formula for the total mechanical energy, we have:

$$E = \gamma m c^2 - GM_{\square} m / r = \gamma m c^2 - a u \quad (8),$$

where we have introduced a new constant $a \equiv GM_{\square} m$.

From (8) we have $\gamma = (E + a u) / m c^2$ and as a consequence, equation (7) becomes

$$u'' + u \left(1 - a^2 / c^2 L^2 \right) = a E / c^2 L^2$$

Returning to initial quantities we find:

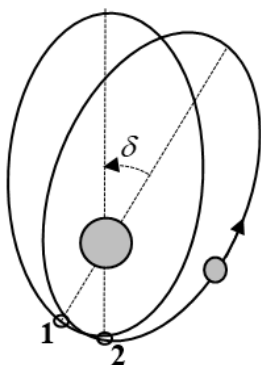


Fig.1 Perihelion shift-axagereted

$$u''(\alpha) + u(\alpha) \left(1 - \frac{G^2 M_{\square}^2 m^2}{c^2 L^2} \right) = \frac{GM_{\square} m E}{c^2 L^2} \quad (9')$$

or

$$u''(\alpha) + \xi^2 u(\alpha) = \frac{GM_{\square} m E}{c^2 L^2} \quad (9)$$

where

$$\xi^2 \equiv 1 - \frac{G^2 M_{\odot}^2 m^2}{c^2 L^2}$$

$$u''(\alpha) = \frac{d^2 u(\alpha)}{d\alpha^2} \quad (10)$$

Equation (9) solves our problem.

Being a nonhomogeneous equation, it has the general solution:

$$u(\alpha) = \frac{EGM_{\odot}m}{\xi^2 c^2 L^2} + C_1 \cos(\alpha\xi) + C_2 (\sin \alpha\xi)$$

$$u(\alpha) = A + B \cos(\alpha\xi) + C \sin(\alpha\xi) \quad (11'),$$

where A, B, C are constants.

For symmetry reasons, we admit a solution which doesn't change with the sign of the azimuthal angle, so we take as physically acceptable the solution:

$$u(\alpha) = A + B \cos(\alpha\xi) \quad (11)$$

This is not a pure harmonic motion because of the small factor ξ . In Solar System, this factor is

very closely equal to 1 because the quantity $\frac{G^2 M_{\odot}^2 m^2}{c^2 L^2} \ll 1$, therefore we can approximate

$$\xi \approx 1 - \frac{G^2 M_{\odot}^2 m^2}{2c^2 L^2} \quad (12)$$

Suppose the first perihelion is situated for $\alpha = 0$ ($\alpha\xi = 0$), which is related to the minimum distance of the planet so $u = \max$. After a full orbit, which means an increase of azimuthal angle by 2π , one must add an amount δ , in order to have the second perihelion, such as to have $\delta = 2\pi(1 - \xi)$. This is the perihelion shift in this approximation (fig.1) (exaggerated in the figure). With ξ from (12) we find the shift for one “planetary year” equal to:

$$\delta = 2\pi(1 - \xi) = \frac{\pi G^2 M_{\odot}^2 m^2}{c^2 L^2} \quad (13)$$

In the case of Mercury, which is the closest planet to the Sun and has a higher velocity, as well as for its nearly circular orbit with a radius R and calculating approximately the angular momentum L from Newtonian law of gravitation we arrive to the formula

$$\delta = \frac{\pi \bar{M}}{R} \quad (14),$$

where $\bar{M} \equiv GM/c^2$ is the mass of the Sun in geometrized units.

We know that the correct formula taken when using General Relativity is 6 times more than this one. Observations on Mercury perihelion shift confirm a shift six times bigger^{2,4,7,8}. In ref.[11] calculation of correct perihelion shift is made using complex functions, as stated in the introduction. Particularly, in [5] the correct perihelion shift is calculated using Runge-Lenz vector.

III. Discussion of perihelion shift in a weak gravitational field making an approximation in a flat three dimensional space.

For a weak gravitational field, which is the case of Solar System, if we modify only the g_{00} component of the metric tensor⁴ and leave the three dimensional space as flat, we can take a Lagrangian of the form

$$\mathcal{L} = c^2 \dot{t}^2 \left(1 + 2\phi_N / c^2\right) - \left(\dot{r}^2 + r^2 \dot{\alpha}^2\right) = c^2 \dot{t}^2 \left(1 - 2GM_\alpha / c^2 r\right) - \left(\dot{r}^2 + r^2 \dot{\alpha}^2\right),$$

where $\phi_N \equiv -GM_\alpha / r$ is the Newtonian gravitational potential, or using the Schwarzschild radius $r_s = 2GM / c^2 = 2\bar{M}$ we have

$$\mathcal{L} = c^2 \dot{t}^2 \left(1 - r_s / r\right) - \left(\dot{r}^2 + r^2 \dot{\alpha}^2\right) \quad (15)$$

The dot expresses the derivative in respect to the *true* time τ (proper time) measured by the watch carried along with the planet during the quasi periodic motion of the planet on the geodesic. Here we use the Newtonian gravitational potential $\phi = -GM_\alpha / r$ and consider the motion in "equatorial plane". This is because, as it is shown below, the angular momentum is conserved. Here the radial coordinate without any doubt is also the *true* distance from the center.

It is easy to find, as usual [2,4,6,9,10], conserved quantities-energy and *angular momentum*. In view of (15), we have $g_{00} = (1 - r_s / r)$; $g_{11} = -1$; $g_{33} = -r^2$ and (3) yields evidently energy and angular momentum:

$$E = mc^2 \dot{t} \left(1 - \frac{r_s}{r}\right); \quad L = mr^2 \dot{\alpha} \quad (16)$$

We could proceed as before using (15,16) and geodesic equation from the Lagrangian, but in this case, it is more convenient to start from the metric (weak field and flat three-dimensional space) of the form:

$$ds^2 = c^2 dt^2 \left(1 - \frac{r_s}{r}\right) - dr^2 - r^2 d\alpha^2 \quad (17)$$

Dividing both sides by $c^2 d\tau^2$, where τ is now a true time measured by a watch carried with the planet we get:

$$1 = \left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{c^2} - \frac{r^2 \dot{\alpha}^2}{c^2}$$

Putting here conserved quantities (16) we get the equation below:

$$c^2 = \frac{E^2}{m^2 c^2 (1 - r_s / r)} - \dot{r}^2 - \frac{L^2}{m^2 r^2}$$

We calculate again the derivative of both sides with respect to τ and get the following equation:

$$0 = -\frac{E^2 r_s \dot{r}}{m^2 c^2 r^2 (1 - r_s / r)^2} - 2 \left(\dot{r}\right) \ddot{r} + \frac{2L^2 \dot{r}}{m^2 r^3}$$

Because we are dealing with elliptic orbits (almost circular orbits) one can divide by the common factor $\dot{r} \neq 0$ (orbits are not exactly circular) and approximating $(1 - r_s / r)^{-2} \approx (1 + 2r_s / r)$ and going again to the new function $u(\alpha) = 1/r(\alpha)$, we get the new equation:

$$0 = -\frac{E^2 r_s u^2 (1 + 2u)}{2m^2 c^2} - \ddot{r} + \frac{L^2 u^3}{m^2} \quad (17')$$

Now we can express $\ddot{r} = -\frac{L^2 u^2 u''}{m^2}$ where the relativistic factor γ is not anymore present, and consequently write the equation which solves our problem:

$$u'' + u \left(1 - \frac{E^2 r_s^2}{c^2 L^2} \right) = \frac{E^2 r_s}{2c^2 L^2} \quad (18)$$

As it is expected, according to the equivalence principle, the mass of the planet is not present. Taking again the constant in parentheses in (18) as

$$\xi^2 = 1 - \frac{E^2 r_s^2}{L^2 c^2} \rightarrow \xi \approx 1 - \frac{E^2 r_s^2}{2L^2 c^2}$$

and reasoning as in the case of the “eclectic” derivation for the perihelion shift we arrive to the formula:

$$\delta = \frac{\pi E^2 r_s^2}{L^2 c^2} \quad (18')$$

Now, we need the ratio E/L which can be taken from (16) to be $E/L = \frac{c^2 dt}{r^2 d\alpha} (1 - r_s/r)$.

When calculating periods of stable circular orbits of planets in the central field of the Sun, we find effectively three periods i.e. the period in respect to the proper time of the planet; the period in respect to the observer situated far away from the center, which corresponds to the period in coordinate time and, finally, the period in respect to the observer, situated in a fixed point of the field⁹. It results that all these three periods are very close to each other, as far as we are dealing with motions in an orbit with a mean distance much greater not only than the Schwarzschild radius but much greater than the distance of the surface of the Sun from the center of the field. For example, for Mercury, the mean distance to the Sun is 5.47×10^{10} m and its velocity is 47.4 km/s, whereas the radius of the Sun is about 7×10^8 m. On the other hand, Sun's Schwarzschild radius is 3km. It is a completely another story when dealing with motions around black holes in which case we can have orbits of radii comparable with the Schwarzschild radius. Thus, we will approximate $E/L \approx c^2 dt / r^2 d\alpha$ and now everything is calculated in coordinate time. Plugging last formula in (18') we get

$$\delta = \frac{\pi r_s^2 c^2}{R^2} \left(\frac{dt}{d\alpha} \right)^2 = \frac{4\pi G^2 M_\alpha^2}{c^2 R^2} \omega^{-2},$$

where R is the mean radius of the orbit. Angular velocity of the planet can be calculated from Newton theory and finally we get the formula for the shift:

$$\delta = \frac{4\pi \bar{M}}{R} \quad (19)$$

This is four times bigger the former result but it is 2/3-d of the correct result taken when we consider not a flat three-dimensional space but a curved one as it is present in Schwarzschild metrics^{2,4,10}. As far as we know, this possible result - $\delta = 4\pi \bar{M} / R$ - hasn't explicitly been reported.

Had we neglected in (17') the quantity $2u$ in respect to 1 i.e. had we taken $2u \ll 1$ than we couldn't have had the shift as it can be demonstrated quiet easily. This is the same as what we could have had, had we have taken what we call quasiNewtonian Lagrangian (in polar coordinates in equatorial plane and for unit mass):

$$\mathcal{L} = -c^2 + \left(\dot{\vec{r}}^2 / 2 \right) + \left(r^2 \dot{\alpha}^2 / 2 \right) + \frac{GM_{\alpha}}{r} \quad (18'')$$

In the latter case we have the energy for unit mass:

$$E = \vec{p} \cdot \vec{v} - \mathcal{L} = c^2 + \left(\dot{\vec{r}}^2 / 2 \right) + \left(r^2 \dot{\alpha}^2 / 2 \right) - GM_{\alpha} / r$$

Up to the *additive* factor c^2 , this is the classical Lagrangian of unit mass in polar coordinates in gravitational field (Kepler problem).

IV. Effective “index of refraction” and bending of light in a weak gravitational field

In the case of light propagation in vacuum, in absence of gravitational forces, as well as in geometrical optics, Fermat principle is fulfilled (an extremum principle)². This principle states that optical path of light rays is extremum and its variation is zero:

$$\delta \int_A^B k_i dx^i = \delta \int_A^B \vec{k} \cdot d\vec{l} = 0 \quad (20),$$

where A, B are two points where light passes and $k_i = \omega \vec{k}_0 / c$ ($i=1,2,3$), where $\vec{k}_0$ is a unit vector-three-dimensional wave vector tangent to the ray, which traces a straight line in a flat space. The frequency of the light is constant, so we have:

$$\delta \int_A^B dl = 0$$

(This is equivalent to the assertion that the strait line which passes through two points in space is the shortest one).

In the case of light propagation in presence of a static gravitational field which is the case of Sun's field, one needs the definition of a covariant wave vector and express the same Fermat principle for calculating the bending of light. Thus, we define a contra variant wave vector k^{μ} ($\mu = 0,1,2,3$), of a norm zero in every coordinate system with zero component equal to $k^0 = \omega / c$ as it is in absence of the field².

In presence of the field, the true distance between two infinitesimally closed points in three-dimensional space is:

$$dl^2 = \gamma_{ik} dx^i dx^k, i,k=1,2,3 \quad (21),$$

where γ_{ik} are spatial metric components which are related to the spatial components of the full four-dimensional metric (static case) according the formula:

$$\gamma_{ik} = -g_{ik}; \gamma^{ik} = -g^{ik}$$

It is easy to show² that the spatial contra variant components of the wave vector are:

$$k^i = \frac{\omega}{c\sqrt{g_{00}}} \frac{dx^i}{dl}$$

Covariant components are:

$$k_i = g_{i\mu} k^{\mu} = -\gamma_{ik} k^k = -\frac{\omega}{c\sqrt{g_{00}}} \gamma_{ik} \frac{dx^k}{dl}$$

Therefore, invariant Fermat principle, from (20) yields:

$$\delta \int_A^B k_i dx^i = \delta \int_A^B -\frac{\omega}{c\sqrt{g_{00}}} \gamma_{ik} \frac{dx^k}{dl} \frac{dx^i}{dl} dl = 0$$

From (21) and cancelling constant quantities we get:

$$\delta \int_A^B k_i dx^i = \delta \int_A^B \frac{1}{\sqrt{g_{00}}} dl = 0 \quad (22)$$

Here, the spatial component dl is a true distance.

Meanwhile, in the case of light propagation in nonhomogeneous continuous media with a real index of refraction $n(x, y, z)$ and geometric optics conditions (light rays) in an Euclidian space we know that the ray is bent in the direction of greater increase of refraction index in space. Fermat principle in this case states that the extremum can be expressed in form³:

$$\delta \int_A^B n(x, y, z) dl = 0 \quad (23)$$

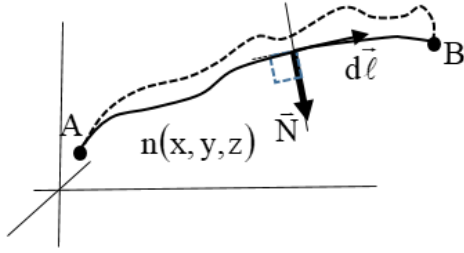


Fig2. Refraction index of the « medium »

We note that we are now proceeding in studying light propagation in a flat three dimensional space where an “effective” refraction index exists changing from point to point of the space.

From the variational principle used for (23) we take the formula which relates the gradient of $n(x, y, z)$ with the curvature radius ρ of the ray³:

$$\frac{1}{\rho} = \frac{\vec{N} \cdot \nabla n}{n} \quad (24)$$

Coming back to (22) we know that metric components vary continuously as functions of spatial coordinates. Comparing with (23) we can state that in a static gravitational field Fermat principle is expressed in the same way as in the case of light propagation in a continuous medium if we take as an effective “index of refraction” equal to:

$$n(x, y, z) = \frac{1}{\sqrt{g_{00}}} \quad (25)$$

Again, in a weak gravitational field we have the modification g_{00} component as shown in the formula:

$$g_{00} = 1 + 2\varphi_N / c^2 \quad (26)$$

where $\varphi_N(r) = -GM/r$ is Newtonian gravitational potential and r is a true distance from the center of the field. With (25,26) we have this index of refraction:

$$n(r) = 1/\sqrt{g_{00}} = (1 - 2\bar{M}/r)^{-1/2} \quad (27)$$

which can be approximated as below:

$$n(r) \approx (1 + \bar{M}/r) \quad (28)$$

This is the index for light propagation in Sun’s weak field, where $\bar{M}$ is Sun’s mass (in geometrized units) and r is the true distance of the photons from its center.

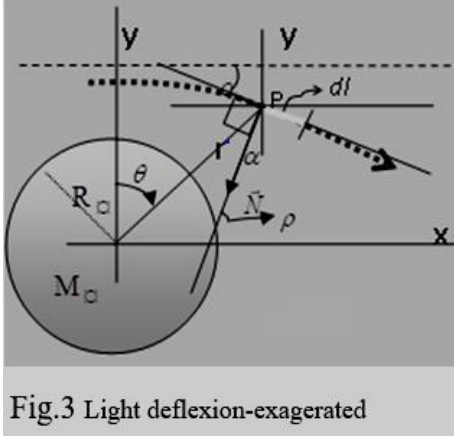
(We note that a similar functional relation exists between an index of true optical refraction and the radial distance r of a point in space from the center of Earth-due to the change of the air density. This refraction is responsible for corrections which must be done for star positions in Astronomy).

In (fig.2) we illustrate the situation of bending of light. $\vec{N}$ is the unit vector perpendicular to the ray at every point. From differential geometry, we know that

$$1/\rho = d\alpha/dl,$$

where ρ is the radius of curvature of the ray at the chosen point P .

We use (24, 28) for solving the problem in the well-known conditions of rays grazing the Sun's disc (Fig.3).



First, we transform (24) as below:

$$\frac{1}{\rho} = \frac{d\alpha}{dl} = \frac{\vec{N} \cdot \nabla n}{n} = \frac{\partial}{\partial N} (\ln n) = \frac{\partial}{\partial N} (\ln[1 + (n-1)])$$

where, in our conditions, the quantity $n-1 \ll 1$.

Expanding the logarithm to first approximation we get:

$$\frac{d\alpha}{dl} \approx \frac{\partial}{\partial N} (n-1) = \frac{\partial}{\partial N} \left(\frac{\bar{M}}{r} \right) \quad (29)$$

The impact parameter of light rays or photons is $R_{\square}$ -Sun's radius and we take into consideration that the deflection of rays is very small.

Geometry considerations lead to these formulas:

$$\cos \alpha = \frac{dx}{dl}; \frac{\partial}{\partial N} = \frac{\partial}{\partial y} \frac{\partial y}{\partial N}; \frac{\partial y}{\partial N} = -\cos \alpha; x \approx R_{\square} \tan \theta; r \approx \frac{R_{\square}}{\cos \theta}; dx \approx \frac{R_{\square} d\theta}{\cos^2 \theta} \quad (30)$$

From (27) we get the following relation for the change of azimuthal angle:

$$d\alpha = \frac{\bar{M}}{R_{\square}} \cos \theta d\theta$$

The total angle accumulated during the whole path is:

$$\delta = \int d\alpha = \int_{-\pi/2}^{\pi/2} \frac{\bar{M}}{R_{\square}} \cos \theta d\theta = \frac{2\bar{M}}{R_{\square}} \equiv \frac{2GM_{\square}}{c^2 R_{\square}} \quad (29)'$$

$$\delta = \frac{2\bar{M}}{R_{\square}} \quad (29)$$

This is half of the correct result taken when one considers the modification of spatial metrics-GR correct result.

From the formula of the approximate interval (17) for radial fall of photons we get in another way the refraction coefficient, which will be useful latter. From (17) we get the coordinate velocity of light to be

$$v_{phot} = \frac{dr}{dt} = c \sqrt{1 - \frac{r_s}{r}} \approx c \left(1 - \frac{r_s}{2r} \right) \quad (30)$$

From here we have $v_{phot} \approx c \left(1 - \frac{r_s}{2r}\right) = \frac{c}{n}$ and the refraction coefficient is the same as before $n(r) = 1 - \frac{r_s}{2r} \approx 1 + \frac{r_s}{2r}$ and $n(r) - 1$ same as (28).

V. Bending of light calculated in the case of curved three-dimensional space in a weak field approximation (“slightly” curved space time)

In this case we have the formula for the interval in the form⁴:

$$ds^2 = \left(1 + \frac{2\varphi(r)}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\varphi(r)}{c^2}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\alpha^2) \quad (31),$$

where $\varphi(r) \equiv -GM_{\odot}/r$ is the Newtonian potential; r as usual is the radial coordinate, such as to have $r = \sqrt{x^2 + y^2 + z^2}$ and r, θ, α are usual spherical coordinates in three-dimensional Euclidian space.

$$dl^2 = (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\alpha^2)$$

For radially falling photons in the field ($d\theta = 0; d\alpha = 0$), in this case of curved three-dimensional space, reasoning same as before we find the light velocity equal to:

$$v_{phot} = \frac{dr}{dt} = c \frac{\left(1 + 2\varphi/c^2\right)^{1/2}}{\left(1 - 2\varphi/c^2\right)^{1/2}} \approx c \left(1 + \varphi/c^2\right)^2 \approx c \left(1 + 2\varphi/c^2\right) = c \left(1 - \frac{2M}{r}\right) \equiv \frac{c}{n(r)} \quad (31),$$

neglecting terms of higher order. From here we find the coefficient

$$n(r) - 1 \approx 2M/r \quad (32)$$

As a result, in this case, we obtain the correct result:

$$\delta = \frac{4M}{R_{\odot}} \quad (33),$$

which is the double of the effect respect to the formula (29)^{4,8,10}.

In the context of PPN formalism, this result is in accord with this approach if one takes the PPN coefficients $\beta, \gamma = 1$ ⁸.

Recently a rather detailed account on this aspect of GR is given in a paper published in International Journal of Astronomy and Astrophysics¹³.

There are some recent attempts⁶ according to which the acceleration of a particle (with mass or massless ones) in free fall is caused by the refraction of its oscillating components by a cloud of exchange particles. These attempts are of course in contrast to general relativity considerations trying to find out another origin of gravitation linked to the structure of elementary particles and their interactions with “a modern ether”. According to their claims

the authors find a coordinate velocity of photons equal to $c \left(1 - \frac{r_s}{r}\right)^p$ where $p = 1$ for radial fall

and $p = 1/2$ for tangential motion of photons. It is clear that the case $p = 1$ corresponds to our result (31).

VI. Conclusions and discussion

1. Where as strict Newtonian approach doesn't yield a perihelion shift, the so called "eclectic" approach corresponding to the Lagrangian (4) yields a perihelion shift of magnitude given by (14) which is the 1/6-th of the correct result given by GR. The time in this later approach is the time in the inertial Solar system.
2. In a weak gravitational field, when making a first approximation and keeping the spatial metric flat, only modification of the g_{00} component of the metric yields for the shift a quantity given by (19), which is 2/3-th of the correct GR result. This possible result hasn't somewhere been reported as far as we know. The time is the proper time of the planet.
3. In a weak field approximation for centro symmetric gravitational field, the field itself can be considered as a « medium » characterized by an effective index of refraction given by (28) and as a consequence we find an amount of bending of light given by (29) which is half of the correct result taken by GR and spectacularly verified in 1919.
4. The correct result of bending of light can be obtained if one consider the approximate metrics in the case of centro symmetric weak field-formula (31) via an effective index of refraction given by (33).
5. Had we proceeded to make an elementary "eclectic" calculation for the bending of light considering gravitational mass of the photons moving in a flat three dimensional space where gravitation of the Sun exists (a calculation found in many textbooks), we would have got half of the result (29) instead of quarter of the respective result of the shift of perihelion. The reason is that in the case of orbits of planets they move all the time at almost same distance from the Sun, while photons come from infinity to a finite distance from the Sun and then again move to infinity, thus not "feeling" all the time a strong gravitational force.

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